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FROBENIUS MORPHISM AND VECTOR BUNDLES ON CYCLES OF PROJECTIVE LINES

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ABSTRACT. In this paper we describe the action of the Frobenius morphism on the indecomposable vector bundles on cycles of projective lines. This gives an answer to a question of Paul Monsky, which appeared in his study of the Hilbert–Kunz theory for plane cubic curves.

This article arose as an answer to a question posed by Paul Monsky in his study of the Hilbert–Kunz theory for plane cubic curves [5]. Let $\boldsymbol{k}$ be an algebraically closed field of characteristic $p > 0$ and E be a projective curve of arithmetic genus one over $\boldsymbol{k}$. We are interested in an explicit description of the action of the Frobenius morphism on the indecomposable vector bundles on E. In the case of elliptic curves, this problem has been solved by Oda [6, Theorem 2.16]. In this article we deal with the case when E is an irreducible rational nodal curve or a cycle of projective lines.

We start by recalling the general technique used in a study of vector bundles on singular projective curves, see [3, 1, 2]. Let X be a reduced singular (projective) curve over $\boldsymbol{k}$, $\pi : \widetilde{X} \to X$ its normalization and $\mathcal{I} := \mathcal{H}om_{\mathcal{O}}\big(\pi_*(\mathcal{O}_{\widetilde{X}}), \mathcal{O}\big) = \mathcal{A}nn_{\mathcal{O}}\big(\pi_*(\mathcal{O}_{\widetilde{X}})/\mathcal{O}\big)$ the conductor ideal sheaf. Denote by $\eta : Z = V(\mathcal{I}) \to X$ the closed artinian subscheme defined by $\mathcal{I}$ (its topological support is precisely the singular locus of X) and by $\tilde{\eta} : \widetilde{Z} \to \widetilde{X}$ its preimage in $\widetilde{X}$, defined by the Cartesian diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{\pi}"'] & \widetilde{X} \arrow[d, "\pi"] \\ Z \arrow[r, "\eta"] & X. \end{tikzcd} \tag{1}$$

Definition 1. The category of triples $\mathsf{Tri}(X)$ is defined as follows.

- Its objects are triples $\big(\widetilde{\mathcal{F}}, \mathcal{V}, \widetilde{\mathsf{m}}\big)$, where $\widetilde{\mathcal{F}} \in \mathsf{VB}(\widetilde{X})$, $\mathcal{V} \in \mathsf{VB}(Z)$ and
$$\mathsf{m} : \tilde{\pi}^*\mathcal{V} \longrightarrow \tilde{\eta}^*\widetilde{\mathcal{F}}$$
is an isomorphism of $\mathcal{O}_{\widetilde{Z}}$–modules, called the *gluing map*.
- The set of morphisms $\mathsf{Hom}_{\mathsf{Tri}(X)}\big((\widetilde{\mathcal{F}}_1, \mathcal{V}_1, \mathsf{m}_1), (\widetilde{\mathcal{F}}_2, \mathcal{V}_2, \mathsf{m}_2)\big)$ consists of all pairs (F, f), where $F : \widetilde{\mathcal{F}}_1 \to \widetilde{\mathcal{F}}_2$ and $f : \mathcal{V}_1 \to \mathcal{V}_2$ are morphisms of vector bundles

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such that the following diagram is commutative

$$\begin{tikzcd} \tilde{\pi}^*\mathcal{V}_1 \arrow[r, "\mathsf{m}_1"] \arrow[d, "\tilde{\pi}^*(f)"'] & \tilde{\eta}^*\widetilde{\mathcal{F}}_1 \arrow[d, "\tilde{\eta}^*(F)"] \\ \tilde{\pi}^*\mathcal{V}_2 \arrow[r, "\mathsf{m}_2"] & \tilde{\eta}^*\widetilde{\mathcal{F}}_2. \end{tikzcd}$$

Theorem 2 (Lemma 2.4 in [3], see also Theorem 1.3 in [2])**.** *Let X be a reduced curve over $\boldsymbol{k}$. Then the functor $\mathbb{F} : \mathsf{VB}(X) \to \mathsf{Tri}(X)$ assigning to a vector bundle $\mathcal{F}$ the triple $(\pi^*\mathcal{F}, \eta^*\mathcal{F}, \mathsf{m}_{\mathcal{F}})$, where $\mathsf{m}_{\mathcal{F}} : \tilde{\pi}^*(\eta^*\mathcal{F}) \to \tilde{\eta}^*(\pi^*\mathcal{F})$ is the canonical isomorphism, is an equivalence of categories.*

Remark 3. In the partial case when X is a configuration of projective lines intersecting transversally, the above theorem also follows from a more general result of Lunts [4].

For a ringed space $(Y, \mathcal{O}_Y)$ over $\boldsymbol{k}$ we denote by φ_Y the Frobenius morphism $(Y, \mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$. Then for an open set $U \subset Y$ the ring homomorphism $\varphi_Y(U) : \mathcal{O}_Y(U) \to \mathcal{O}_Y(U)$ is given by the formula $\varphi_Y(f) = f^p$, $f \in \mathcal{O}_Y(U)$. For simplicity, we shall frequently omit the subscript in the notation of the Frobenius map.

Definition 4. Let endofunctor $\mathbb{P} : \mathsf{Tri}(X) \longrightarrow \mathsf{Tri}(X)$ be defined as follows. For an object $\mathcal{T} = (\widetilde{\mathcal{F}}, \mathcal{V}, \mathsf{m})$ of the category $\mathsf{Tri}(X)$ we set $\mathbb{P}(\mathcal{T}) := \bigl(\varphi^*\widetilde{\mathcal{F}}, \varphi^*\mathcal{V}, \mathsf{m}^{\varphi}\bigr)$, where the gluing map m^{φ} is determined via the commutative diagram

$$\begin{tikzcd} \varphi^*\tilde{\pi}^*\mathcal{V} \arrow[r, "\varphi^*(\mathsf{m})"] \arrow[d, "\mathsf{can}"'] & \varphi^*\tilde{\eta}^*\widetilde{\mathcal{F}} \arrow[d, "\mathsf{can}"] \\ \tilde{\pi}^*\varphi^*\mathcal{V} \arrow[r, "\mathsf{m}^{\varphi}"] & \tilde{\eta}^*\varphi^*\widetilde{\mathcal{F}}, \end{tikzcd}$$

and both vertical maps are canonical isomorphisms.

Lemma 5. *Consider the following diagram of categories and functors:*

$$\begin{tikzcd} \mathsf{VB}(X) \arrow[r, "\mathbb{F}"] \arrow[d, "\varphi_X^*"'] & \mathsf{Tri}(X) \arrow[d, "\mathbb{P}"] \\ \mathsf{VB}(X) \arrow[r, "\mathbb{F}"] & \mathsf{Tri}(X), \end{tikzcd}$$

Then there exists an isomorphism $\mathbb{P} \circ \mathbb{F} \to \mathbb{F} \circ \varphi_X^*$.

Proof. Let $\mathcal{F}$ be a vector bundle on X. Then the canonical isomorphisms $\varphi^*\tilde{\eta}^*\mathcal{F} \to \tilde{\eta}^*\varphi^*\mathcal{F}$ and $\varphi^*\pi^*\mathcal{F} \to \pi^*\varphi^*\mathcal{F}$ induce the commutative diagram

$$\begin{tikzcd} \tilde{\pi}^*\varphi^*\tilde{\eta}^*\mathcal{F} \arrow[r, "\mathsf{m}^{\varphi}_{\mathcal{F}}"] \arrow[d, "\mathsf{can}"'] & \tilde{\eta}^*\varphi^*\pi^*\mathcal{F} \arrow[d, "\mathsf{can}"] \\ \tilde{\pi}^*\tilde{\eta}^*\varphi^*\mathcal{F} \arrow[r, "\mathsf{m}_{\varphi^*\mathcal{F}}"] & \tilde{\eta}^*\pi^*\varphi^*\mathcal{F}, \end{tikzcd}$$

which yields the desired isomorphism of functors. □

Next, we need a description of the action of the Frobenius map on the vector bundles on a projective line. Let (z_0, z_1) be coordinates on $V = \mathbb{C}^2$. They induce homogeneous coordinates $(z_0 : z_1)$ on $\mathbb{P}^1 = \mathbb{P}^1(V) = \big(V \setminus \{0\}\big)/\sim$, where $v \sim \lambda v$ for all $v \in V$ and $\lambda \in \mathbb{C}^*$. We set $U_0 = \big\{(z_0 : z_1) | z_0 \neq 0\big\}$ and $U_\infty = \big\{(z_0 : z_1) | z_1 \neq 0\big\}$ and put $0 := (1 : 0)$, $\infty := (0 : 1)$, $z = z_1/z_0$ and $w = z_0/z_1$. So, z is a coordinate in a neighbourhood of 0. If $U = U_0 \cap U_\infty$ and $w = 1/z$ is used as a coordinate on U_∞, then the transition function of the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ is

$$U_0 \times \mathbb{C} \supset U \times \mathbb{C} \xrightarrow{(z,v)\mapsto\left(\frac{1}{z}, \frac{v}{z^n}\right)} U \times \mathbb{C} \subset U_\infty \times \mathbb{C}. \tag{2}$$

The proof of the following lemma is straightforward.

Lemma 6. *For any $n \in \mathbb{Z}$ we have: $\varphi^*\big(\mathcal{O}_{\mathbb{P}^1}(n)\big) \cong \mathcal{O}_{\mathbb{P}^1}(np)$.*

Next, recall the following classical result on vector bundles on a projective line.

Theorem 7 (Birkhoff–Grothendieck)**.** *Any vector bundle $\widetilde{\mathcal{F}}$ on $\mathbb{P}^1$ splits into a direct sum of line bundles:*

$$\widetilde{\mathcal{F}} \cong \bigoplus_{l \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{m_l}. \tag{3}$$

Now assume that E is an *irreducible* rational *nodal* curve. Note that by the definition of being nodal we have: $Z = \mathsf{Spec}(\boldsymbol{k})$.

Example 8. The plane cubic curve $E \subset \mathbb{P}^2$, given by the homogeneous equation $x^3 + y^3 - xyz = 0$, is an irreducible rational curve with a nodal singularity at $(0 : 0 : 1)$.

Theorem 7 implies that for an object $(\widetilde{\mathcal{F}}, \mathcal{V}, \widetilde{\mathsf{m}})$ of $\mathsf{Tri}(E)$ with $\mathrm{rk}(\widetilde{\mathcal{F}}) = n$, we have

$$\widetilde{\mathcal{F}} = \bigoplus_{l \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{m_l} \quad \text{and} \quad \mathcal{V} \cong \mathcal{O}_Z^n, \quad \text{where} \sum_{l \in \mathbb{Z}} m_l = n.$$

The vector bundle $\mathcal{V}$ is free, because Z is artinian. In order to describe the morphism $\widetilde{\mathsf{m}}$ in the terms of matrices, some additional choices have to be made.

Recall that the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ is isomorphic to the sheaf of sections of the so-called tautological line bundle

$$\big\{(l, v) | \, v \in l\big\} \subset \mathbb{P}^1(V) \times V = \mathcal{O}^2_{\mathbb{P}^1}.$$

The choice of coordinates on $\mathbb{P}^1$ fixes two distinguished elements, z_0 and z_1, in the vector space $\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}\big)$:

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{C}^2 \arrow[dr] & \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[l, hook'] \arrow[r, "z_i"] \arrow[d] & \mathbb{P}^1 \times \mathbb{C} \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

where z_i maps $\big(l, (v_0, v_1)\big)$ to (l, v_i) for $i = 0, 1$. It is clear that the section z_0 vanishes at ∞ and z_1 vanishes at 0. In what follows, we shall assume that the coordinates on the normalization $\widetilde{E}$ are chosen in such a way that $\mathsf{Spec}(\boldsymbol{k} \times \boldsymbol{k}) \cong \widetilde{Z} = \pi^{-1}(Z) = \big\{0, \infty\big\}$.

Definition 9. For any $l \in \mathbb{Z}$ we define the isomorphism $\xi_l : \tilde{\eta}^*\big(\mathcal{O}_{\mathbb{P}^1}(l)\big) \to \mathcal{O}_{\widetilde{Z}}$ by the formula $\xi_l(s) = \big(\frac{s}{z_0^l}(0), \frac{s}{z_1^l}(\infty)\big)$, where s is an arbitrary local section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(l)$. Hence, for any vector bundle $\widetilde{\mathcal{F}}$ of rank n on $\mathbb{P}^1$ given by the formula (3), we have the induced isomorphism $\xi_{\widetilde{\mathcal{F}}} : \tilde{\eta}^*\widetilde{\mathcal{F}} \to \mathcal{O}_{\widetilde{Z}}^n$.

Let $(\widetilde{\mathcal{F}}, \mathcal{O}_Z^n, \mathsf{m})$ be an object in the category of triples $\mathsf{Tri}(E)$. Note that we have a unique morphism $M(\mathsf{m})$ making the following diagram commutative:

$$
\begin{tikzcd}
\tilde{\pi}^*\mathcal{O}_Z^n \arrow[r, "\mathsf{m}"] \arrow[d, "\mathsf{can}"'] & \tilde{\eta}^*\widetilde{\mathcal{F}} \arrow[d, "\xi_{\widetilde{\mathcal{F}}}"] \\
\mathcal{O}_{\widetilde{Z}}^n \arrow[r, "M(\mathsf{m})"] & \mathcal{O}_{\widetilde{Z}}^n,
\end{tikzcd}
\tag{4}
$$

where the first vertical map is the canonical isomorphism. Moreover, $M(\mathsf{m})$ is given by a pair of invertible $(n \times n)$ matrices $M(0)$ and $M(\infty)$ over the field $\boldsymbol{k}$. Applying to (4) the functor φ^*, we get the following commutative diagram:

$$
\begin{tikzcd}
\tilde{\pi}^*\varphi^*\mathcal{O}_Z^n \arrow[r, "\mathsf{m}^\varphi"] \arrow[ddd, bend right=60, "\mathsf{can}"'] & \tilde{\eta}^*\varphi^*\widetilde{\mathcal{F}} \arrow[ddd, bend left=60, "\xi_{\varphi^*\widetilde{\mathcal{F}}}"] \\
\varphi^*\tilde{\pi}^*\mathcal{O}_Z^n \arrow[r, "\varphi^*(\mathsf{m})"] \arrow[u, "\mathsf{can}"] \arrow[d, "\mathsf{can}"'] & \varphi^*\tilde{\eta}^*\widetilde{\mathcal{F}} \arrow[u, "\mathsf{can}"'] \arrow[d, "\varphi^*(\xi_{\widetilde{\mathcal{F}}})"] \\
\varphi^*\big(\mathcal{O}_{\widetilde{Z}}^n\big) \arrow[r, "\varphi^*(M(\mathsf{m}))"] \arrow[d, "\mathsf{can}"'] & \varphi^*\big(\mathcal{O}_{\widetilde{Z}}^n\big) \arrow[d, "\mathsf{can}"] \\
\mathcal{O}_{\widetilde{Z}}^n \arrow[r, "M(\mathsf{m}^\varphi)"] & \mathcal{O}_{\widetilde{Z}}^n.
\end{tikzcd}
\tag{5}
$$

Corollary 10. Let E be an irreducible rational nodal curve over a field $\boldsymbol{k}$ of characteristic $p > 0$ and $\mathcal{F}$ be a vector bundle corresponding to the triple $(\widetilde{\mathcal{F}}, \mathcal{O}_Z^n, \mathsf{m})$, where $\widetilde{\mathcal{F}} \cong \oplus_{l\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^1}(l)^{m_l}$ and m is given by a pair of matrices

$$
M(0) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \text{and} \quad M(\infty) = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}.
$$

Then the vector bundle $\varphi^*\mathcal{F}$ is given by the triple $(\varphi^*\widetilde{\mathcal{F}}, \mathcal{O}_Z^n, \mathsf{m}^\varphi)$, where $\varphi^*\widetilde{\mathcal{F}} \cong \oplus_{l\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^1}(lp)^{m_l}$ and m^φ corresponds to the pair of matrices

$$
\begin{pmatrix} a_{11}^p & a_{12}^p & \dots & a_{1n}^p \\ a_{21}^p & a_{22}^p & \dots & a_{2n}^p \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^p & a_{n2}^p & \dots & a_{nn}^p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{11}^p & b_{12}^p & \dots & b_{1n}^p \\ b_{21}^p & b_{22}^p & \dots & b_{2n}^p \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}^p & b_{n2}^p & \dots & b_{nn}^p \end{pmatrix}.
$$

Remark 11. Recall, that the indecomposable vector bundles on an irreducible nodal rational curve E over an algebraically closed field $\boldsymbol{k}$ are described by the following data:

- a non-periodic sequence of integers $\mathbb{d} = (d_1, \dots, d_l)$,
- a positive integer m,
- a continuous parameter $\lambda \in \boldsymbol{k}^*$,

see [3, Theorem 2.12] or [1, Section 3]. The corresponding indecomposable vector bundle $\mathcal{F} = \mathcal{B}\big(\mathbb{d}, m, \lambda\big)$ has rank lm. By the definition (see e.g. [1, Algorithm 1]) the corresponding triple $\mathbb{P}(\mathcal{F}) \cong \Big(\widetilde{\mathcal{F}}, \mathcal{V}, \big(M(0), M(\infty)\big)\Big)$ is the following: $\widetilde{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^1}(d_1)^m \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(d_l)^m$, $\mathcal{V} = \mathcal{O}_Z^{lm}$ and the gluing matrices are

$$M(0) = \mathbb{1}_{ml \times ml} \quad \text{and} \quad M(\infty) = \begin{pmatrix} 0 & \mathbb{I} & 0 & \dots & 0 \\ 0 & 0 & \mathbb{I} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \mathbb{I} \\ \mathbb{J} & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $\mathbb{I} = \mathbb{1}_{m \times m}$ is the identity matrix of size m and $\mathbb{J}$ is the Jordan block $J_m(\lambda)$. $\square$

Theorem 12. *Let E be an irreducible nodal rational curve over an algebraically closed field $\boldsymbol{k}$ of characteristic $p > 0$, $\mathbb{d} = (d_1, d_2, \dots, d_l)$ be a non-periodic sequence of integers, $m \in \mathbb{N}$ and $\lambda \in \boldsymbol{k}^*$. Let $\mathcal{F} = \mathcal{B}\big(\mathbb{d}, m, \lambda\big)$ be the corresponding indecomposable vector bundle on E. Then we have:*

$$\varphi^* \mathcal{F} \cong \mathcal{B}\big((pd_1, pd_2, \dots, pd_l), m, \lambda^p\big). \tag{6}$$

In particular, the vector bundle $\varphi^* \mathcal{F}$ is indecomposable.

Proof. It is a direct consequence of Theorem 2, Corollary 10 and Remark 11. $\square$

Remark 13. The same argument literally applies to the case, when E is a cycle of projective lines. In particular, an analogous formula (6) holds in that case, too. Note that in the case of elliptic curves it is in general *not true* that the pull-back of an indecomposable vector bundle under the Frobenius morphism is again indecomposable [6, Theorem 2.16].

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